NONSTATIONARY EQUATIONS OF NONLINEAR ELASTICITY THEORY IN EULERIAN COORDINATES

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In this paper we study a system of differential equations which describes nonstationary processes of nonlinear elasticity theory in an isotropic medium. Such a medium is characterized by an equation of state $E=E(I_1, I_2, I_3, S)$, expressing the internal energy density E (per unit mass) on the deformation tensor invariants I_1, I_2, I_3 and on the entropy S.

The "viscoelastic" Kelvin–Maxwell terms are included in the equations in order to describe plastic deformation processes. It is necessary to introduce these terms in order not to violate the continuity equations. The satisfaction of the law of entropy increase during viscoelastic processes imposes restrictions on the equation of state in the form of certain inequalities. We show, in what follows, that these inequalities are always satisfied if the equation of state $E=E(I_1, I_2, I_3, S)$ is such that the system of equations is hyperbolic. The main content of our paper is the study of thermodynamic identities, the composition of the characteristic equation, the formulation of the conditions for hyperbolicity, and the reduction of the system of equations to the symmetric hyperbolic form due to Friedrichs. We have been unable to find in the literature a system of equations written in a form satisfying these requirements. Therefore the first items in our paper are concerned with a justification of this form.*

The invariants ρ, D, and Δ of the deformation tensor, which are used in writing out the equations of the characteristic cone in a system of coordinates attached to the principal axes of the deformation tensor (Sec. 5), make it possible to see, when $E(\rho, D, \Delta, S)$ does not depend on D and Δ, that the characteristics of the system in question reduce to the characteristics of the hydrodynamic equations.

1. Conservation Laws and Murnaghan's Formulas. We assume that the state of a continuous medium is characterized by the distribution of the density ρ, the internal energy density ρE, the entropy density ρS, the velocity vector fields with components u_i ($i=1, 2, 3$), and the stress tensor with components σ_{ik} ($\sigma_{ik}=\sigma_{ki}$, $i, k=1, 2, 3$). The medium as it moves must then satisfy laws for the conservation of mass, momentum, energy, and entropy. We omit any consideration of heat transfer processes. Terms which describe entropy growth during relaxational viscoelastic processes will be taken up later. The conservation laws for the case in question, written in differential form, reduce to the following divergence equations:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_k}{\partial x_k}=0$$

$$\frac{\partial \rho u_i}{\partial t}+\frac{\partial(\rho u_i u_k-\sigma_{ik})}{\partial x_k}=0$$

$$\frac{\partial \rho(E+{}^1/_2 u_i u_i)}{\partial t}+\frac{\partial[\rho u_k(E+{}^1/_2 u_i u_i)-u_i\sigma_{ik}]}{\partial x_k}=0$$

$$\frac{\partial \rho S}{\partial t}+\frac{\partial \rho S u_k}{\partial x_k}=0$$

To describe the state of the medium it is also necessary to include the deformation tensor ε_{ik} or the metric tensor associated with it, $g_{ik}=\delta_{ik}-2\varepsilon_{ik}$. Using the first principle of thermodynamics, we can,

*L. I. Sedov has called attention to [1], wherein a study is made of the complete system of linear equations for small perturbations in a viscoelastic liquid.

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through the use of the equation of state, relate the stress and deformation tensors by means of Murnaghan's formulas (see [2-4])

$$\sigma_{ik} = \rho \left(\frac{\partial E}{\partial \varepsilon_{ik}} - 2\varepsilon_{i\alpha} \frac{\partial E}{\partial \varepsilon_{\alpha k}} \right)$$
$$\rho = \rho_0 \sqrt{\det \| \delta_{ik} - 2\varepsilon_{ik} \|}$$
$$E = E(I_1, I_2, I_3, S)$$

Here ρ and E are assumed to be known functions of the deformation tensor (ρ_0 is the density of the undeformed medium) and the I_k are the invariants of the deformation tensor

$$I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$
$$I_2 = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{vmatrix} + \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{32} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{33} & \varepsilon_{31} \\ \varepsilon_{13} & \varepsilon_{11} \end{vmatrix}$$
$$I_3 = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{vmatrix}$$

Thus E is a symmetric function of the characteristic values of the deformation tensor. For a function of the deformation tensor it is necessary to distinguish differentiation with respect to the components ε_{ij} and ε_{ji}, otherwise, after differentiating, we would have twice as many quantities.

Subject to these assumptions concerning the equation of state, the stress tensor σ_{ik} turns out be symmetric.

It is evident that Murnaghan's formulas are tensorial in nature (in the present notation this is only true relative to orthogonal transformations of the coordinate system). If the axes of the coordinate system coincide with the principal directions of the deformation (also of the stress) tensor, then

$$\varepsilon_{ik} = 0 \quad (i \neq k), \qquad \varepsilon_{ii} \equiv \varepsilon_i$$
$$\sigma_{ik} = 0 \quad (i \neq k), \qquad \sigma_{ii} = \rho (1 - 2\varepsilon_i) \frac{\partial E}{\partial \varepsilon_i} = \rho \frac{1}{\sqrt{1 - 2\varepsilon_i}} \frac{\partial E}{\partial [(1 - 2\varepsilon_i)^{-1/2}]} \equiv \sigma_i$$
$$(\rho = \rho_0 \sqrt{(1 - 2\varepsilon_1)(1 - 2\varepsilon_2)(1 - 2\varepsilon_3)})$$

For the parametrization $E = E(a_1, a_2, a_3, S)$ of the equation of state, where $a_i = (1 - 2\varepsilon_i)^{-1/2}$, the Murnaghan formulas reduce to the following form:

$$\sigma_i = \rho a_i E a_i$$

or, taking note of the fact that $\rho = \rho_0 / a_1 a_2 a_3$

$$\sigma_1 = \frac{\rho_0}{a_2 a_3} E_{a_1}(a_1, a_2, a_3, S)$$
$$\sigma_2 = \frac{\rho_0}{a_3 a_1} E_{a_2}(a_1, a_2, a_3, S)$$
$$\sigma_3 = \frac{\rho_0}{a_1 a_2} E_{a_3}(a_1, a_2, a_3, S)$$

Use of the parameters a_1, a_2, a_3 makes it possible to give an intuitive derivation of Murnaghan's formulas for the stress tensor. If a volume of the medium in its undeformed and unstressed state occupies a rectangular parallelepiped with edges $(\Delta x_1)_0$, $(\Delta x_2)_0$, $(\Delta x_3)_0$, then, after an adiabatic deformation, it occupies a parallelepiped with the edges $\Delta x_i = a_i \times (\Delta x_i)_0$, i.e., a_1, a_2, a_3 give the expansion coefficients along the axes. The forces acting on the faces of the parallelepiped are as follows:

$$F_1 = \sigma_1 \Delta x_2 \Delta x_3 = \sigma_1 a_2 a_3 (\Delta x_2)_0 (\Delta x_3)_0$$
$$F_2 = \sigma_2 \Delta x_3 \Delta x_1 = \sigma_2 a_3 a_1 (\Delta x_3)_0 (\Delta x_1)_0$$
$$F_3 = \sigma_3 \Delta x_1 \Delta x_2 = \sigma_3 a_1 a_2 (\Delta x_1)_0 (\Delta x_2)_0$$

Subjet to the variations δa_i of the parameters a_i, these forces produce through the displacements a_i an amount of work equal to

$$F_1 \delta a_1 (\Delta x_1)_0 + F_2 \delta a_2 (\Delta x_2)_0 + F_3 \delta a_3 (\Delta x_3)_0 = (\sigma_1 \delta a_1 a_2 a_3 + \sigma_2 a_1 \delta a_2 a_3 + \sigma_3 a_1 a_2 \delta a_3)(\Delta x_1)_0 (\Delta x_2)_0 (\Delta x_3)_0$$

which goes to increase the internal energy

$$(\rho\Delta x_1\Delta x_2\Delta x_3)\,E\,(a_1, a_2, a_3, S) = \rho_0\,(\Delta x_1)_0\,(\Delta x_2)_0\,(\Delta x_3)_0 E\,(a_1, a_2, a_3, S)$$

$$\rho_0\,(\Delta x_1)_0\,(\Delta x_2)_0\,(\Delta x_3)_0\,\delta E = a_1 a_2 a_3\left(\sigma_1\frac{\delta a_1}{a_1} + \sigma_2\frac{\delta a_2}{a_2} + \sigma_3\frac{\delta a_3}{a_3}\right)\times$$
$$\times\,(\Delta x_1)_0\,(\Delta x_2)_0\,(\Delta x_3)_0$$

Cancelling the $(\Delta x_1)_0$ $(\Delta x_2)_0$ $(\Delta x_3)_0$, we arrive at Murnaghan's formulas

$$\rho_0\delta E = \sigma_1 a_2 a_3\delta a_1 + \sigma_2 a_1 a_3\delta a_2 + \sigma_3 a_1 a_2\delta a_3$$
$$\sigma_1 = \frac{\rho_0}{a_2 a_3}\frac{\partial E}{\partial a_1}, \quad \sigma_2 = \frac{\rho_0}{a_3 a_1}\frac{\partial E}{\partial a_2}, \quad \sigma_3 = \frac{\rho_0}{a_1 a_2}\frac{\partial E}{\partial a_3}$$

As usual, E_S has the meaning of temperature; therefore, in the parameters a_1, a_2, a_3, and S, we have the following thermodynamic identity:

$$\delta E\,(a_1, a_2, a_3, S) = \frac{a_2 a_3}{\rho_0}\sigma_1\delta a_1 + \frac{a_3 a_1}{\rho_0}\sigma_2\delta a_2 + \frac{a_1 a_2}{\rho_0}\sigma_3\delta a_3 + T\delta S$$

In the sequel, it is occasionally convenient to use, instead of the parameters a_i characterizing the principal axes of the deformation tensor, the parameters

$$\rho = \frac{\rho_0}{a_1 a_2 a_3}, \qquad d_i = \ln\frac{a_i}{\sqrt[3]{a_1 a_2 a_3}}$$

and also the invariants ρ

$$D = {}^1/_2\,(d_1^2 + d_2^2 + d_3^2), \;\; \Delta = d_1 d_2 d_3$$

The invariant ρ (density) characterizes the degree of compression of an elementary volume during deformation while the invariants D and Δ characterize its change of shape. In the case of small deformations the invariant D coincides with the quadratic invariant of the deviator of the deformation tensor

$$D = {}^1/_2\,[(\varepsilon_1 - {}^1/_3\,(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))^2 + (\varepsilon_2 - {}^1/_3\,(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))^2 +$$
$$+ (\varepsilon_3 - {}^1/_3\,(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))^2] + O\,(|\,\varepsilon_1\,|^3 + |\,\varepsilon_2\,|^3 + |\,\varepsilon_3\,|^3)$$

For small deformations the quantity Δ is a quantity of the third order of smallness, and in linear elasticity theory the dependence of $E\,(\rho,\ D,\ \Delta S)$ on Δ is not considered.

For reference we give the expressions for the principal stresses, which follow from Murnaghan's formulas, wherein we assume the parametrization $E(\rho,\ D, \Delta,\ S)$

$$\begin{aligned}\sigma_1 &= -\rho^2 E_\rho + \rho d_1 E_D - {}^1/_3\,\rho\,(d_2^2 + d_3^2 - 2d_1^2)\,E_\Delta\\ \sigma_2 &= -\rho^2 E_\rho + \rho d_2 E_D - {}^1/_3\,\rho\,(d_3^2 + d_1^2 - 2d_2^2)\,E_\Delta\\ \sigma_3 &= -\rho^2 E_\rho + \rho d_3 E_D - {}^1/_3\,\rho\,(d_1^2 + d_2^2 - 2d_3^2)\,E_\Delta\end{aligned} \tag{1.1}$$

It follows from these formulas, in particular, that the so-called mean pressure

$$-{}^1/_3\,(\sigma_{11} + \sigma_{22} + \sigma_{33}) = -{}^1/_3\,(\sigma_1 + \sigma_2 + \sigma_3) = \rho^2 E_\rho\,(\rho, D, \Delta, S)$$

may be calculated from the usual formula $p = \rho^2\ E_\rho\ (\rho,\ S)$ for the pressure in a gas.

The values d_1, d_2, d_3 can (to within their order) be determined from the invariants D and Δ as the roots of the cubic equation

$$d^3 - Dd - \Delta = 0$$

It is well known that all the roots of this equation will be real if and only if the inequality

$$(D\,/\,3)^3 \geqslant (\Delta\,/\,2)^2$$

is satisfied.

If ρ, d_1, d_2, d_3 are known, then any other parameters defining a deformation may be calculated from them. Thus, for example,

$$a_i = e^{d_i}\sqrt[3]{\rho_0\,/\,\rho}, \quad \varepsilon_i = {}^1/_2\,[1 - e^{-2d_i}\,(\rho\,/\,\rho_0)^{2/3}]$$

We assume the equation of state $E = E\ (\rho,\ D,\ \Delta,\ S)$ to be such that corresponding to given principal stresses σ_1, σ_2, σ_3 the corresponding principal values ε_1, ε_2, ε_3 of the deformation tensor can be re-established with the help of the Murnaghan formulas (1.1). This assumption is needed to determine the unstressed "initial state" for each element of the deforming medium relative to which the deformation is calculated (see [5], p. 67).

2. Equations Describing Time Variation of the Deformation Tensor. At t = 0 let the metric deformation tensor $g_{ik}^\circ = \delta_{ik} - 2\ \varepsilon_{ik}$ define the deformation which must be produced in the neighborhood of each point in order that the stress tensor at the point be zero. The tensor g_{ik}° defines a metric (element of length)

$$dS^2 = g_{ik}^\circ dx^{\circ i} dx^{\circ k}$$

A moving material point with coordinates x^i at time t will have coordinates $x^{\circ i}$ at the time t = 0,

$$x^{\circ i} = x^{\circ i}(x^1, x^2, x^3, t)$$

Since along the trajectory of a material point the initial coordinate of the point is constant, we have

$$\frac{dx^{\circ i}}{dt} \equiv \frac{\partial x^{\circ i}}{\partial t} + u^k \frac{\partial x^{\circ i}}{\partial x^k} = 0$$

Differentiating this equation with respect to x^j, we obtain

$$\frac{d}{dt}\left(\frac{\partial x^{\circ i}}{\partial x^j}\right) + \frac{\partial u^k}{\partial x^j}\frac{\partial x^{\circ i}}{\partial x^k} = 0$$

We determine the metric tensor $g_{ik} = \delta_{ik} - 2\,\varepsilon_{ik}$ in such a way that the element of length dS of a given moving material vector stays the same. We obtain

$$dS^2 = g_{ik}\, dx^i\, dx^k = g_{ik}^\circ\, dx^{\circ i}\, dx^{\circ k}$$

Therefore

$$g_{ij} = \overset{\circ}{g}_{\alpha\beta} \frac{\partial x^{\circ\alpha}}{\partial x^i}\frac{\partial x^{\circ\beta}}{\partial x^j}$$

and, consequently,

$$\frac{dg_{ij}}{dt} = \overset{\circ}{g}_{\alpha\beta}\frac{d}{dt}\left(\frac{\partial x^{\circ\alpha}}{\partial x^i}\right)\frac{\partial x^{\circ\beta}}{\partial x^j} + \overset{\circ}{g}_{\alpha\beta}\frac{\partial x^{\circ\alpha}}{\partial x^i}\frac{d}{dt}\left(\frac{\partial x^{\circ\beta}}{\partial x^j}\right) = -\overset{\circ}{g}_{\alpha\beta}\frac{\partial x^{\circ\alpha}}{\partial x^k}\frac{\partial x^{\circ\beta}}{\partial x^j}\frac{\partial u^k}{\partial x^i} - \overset{\circ}{g}_{\alpha\beta}\frac{\partial x^{\circ\alpha}}{\partial x^i}\frac{\partial x^{\circ\beta}}{\partial x^k}\frac{\partial u^k}{\partial x^j}$$

Thus

$$\frac{dg_{ij}}{dt} = -g_{i\alpha}\frac{\partial u^\alpha}{\partial x^j} - g_{j\alpha}\frac{\partial u^\alpha}{\partial x^i}$$

We note that since the behavior of the medium is being described in an orthogonal cartesian coordinate system x_1, x_2, x_3, then $x^i = x_i$, $u^i = u_i$. Now, using the definition of the deformation tensor $\varepsilon_{ij} = 1/2\,(\delta_{ij} - g_{ij})$, we obtain equations for the time variation of the deformation tensor components

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}(\delta_{i\alpha} - 2\varepsilon_{i\alpha})\frac{\partial u_\alpha}{\partial x_j} + \frac{1}{2}(\delta_{j\alpha} - 2\varepsilon_{j\alpha})\frac{\partial u_\alpha}{\partial x_i} \tag{2.1}$$

More detailed information concerning a kinematically deforming medium is available in L. I. Sedov's books (see, for example, Eqs. (6.12) on p. 120 of [6]).

We now explain through an example involving small deformations a method of introducing into the equation terms describing the relaxation of tangential stresses. Later on, we assume an extension of this method to finite deformations.

In the case of small deformations ($\varepsilon_{ij} \ll 1$) the Eqs. (2.1) assume a form well known in linear elasticity theory

$$\frac{d\varepsilon_{ij}}{dt} - \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0$$

For small deformations we can take the equation of state in the simplified form

$$\rho_0 E = {}^1/_2\,\lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 + \mu\,(\varepsilon_{ij}\varepsilon_{ji})$$

from which it follows that

$$\sigma_{ij} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\,\delta_{ij} + 2\mu\varepsilon_{ij} \tag{2.2}$$

and, consequently, that

$$\frac{d\sigma_{ij}}{dt} = \lambda \frac{\partial u_k}{\partial x_k}\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

In a viscoelastic medium, for example, in metals in the presence of sufficiently large shear stresses, it is necessary to introduce into the equations for $d\sigma_{ij}/dt$ terms proposed by Kelvin, Maxwell, Voight, and others, which describe relaxation of the stress deviator. After the introduction of these terms, the equations in the linear case can be written as

$$\frac{d\sigma_{ij}}{dt} = \lambda \frac{\partial u_k}{\partial x_k}\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\sigma_{ij} - {}^1/_3\,(\sigma_{11} + \sigma_{22} + \sigma_{33})\,\delta_{ij}}{\tau}$$

Since the σ_{ij} and the ε_{ij} are related through the Eqs. (2.2), the equations we have written out for the σ_{ij} are equivalent to the following equations for the ε_{ij}:

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] - \frac{\varepsilon_{ij} - {}^1/_3\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\,\delta_{ij}}{\tau}$$

Here τ is the characteristic relaxation time of the tangential stresses. In these equations the terms in the square brackets describe the variation of the deformation tensor with motion of the medium; the remaining terms in the right-hand sides of these equations give a phenomenonological account of the time variation of the "initial" state relative to which the deformation tensor is calculated. There arises the natural desire, even in the case of nonlinear viscoelasticity, to write

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}(\delta_{i\alpha} - 2\varepsilon_{i\alpha})\frac{\partial u_\alpha}{\partial x_j} + \frac{1}{2}(\delta_{j\alpha} - 2\varepsilon_{j\alpha})\frac{\partial u_\alpha}{\partial x_i} - \frac{\varepsilon_{ij} - {}^1/_3\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\,\delta_{ij}}{\tau}$$

Here, however, an important circumstance exists which determines the method for writing down the relaxational terms. When relaxational terms of the type

$$\varphi_{ij} = -\frac{\varepsilon_{ij} - {}^1/_3\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})}{\tau}$$

are not included in the right member of the equations for the ε_{ij}, the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0$$

then follows from the Eqs. (2.1).

The density ρ may be expressed in terms of the metric tensor g_{ij} and the deformation tensor ε_{ij} through the formulas

$$\rho = \rho_0 \sqrt{\det\|g_{ij}\|} = \rho_0 \sqrt{\det\|\delta_{ij} - 2\varepsilon_{ij}\|}$$

From this it follows that

$$g_{i\alpha}\rho_{\varepsilon_{\alpha j}} = -\rho\delta_{ij}$$

i.e.,

$$(\delta_{i\alpha} - 2\varepsilon_{i\alpha})\,\rho_{\varepsilon_{\alpha j}} = -\rho\delta_{ij}, \quad (\delta_{\alpha i} - 2\varepsilon_{\alpha i})\,\rho_{\varepsilon_{j\alpha}} = -\rho\delta_{ij}$$

We now multiply each of the Eqs. (2.1) by the corresponding $\rho_{\varepsilon_{ij}}$ and add to get

$$0 = \frac{d\rho}{dt} - \frac{1}{2}\rho_{\varepsilon_{ij}}(\delta_{i\alpha} - 2\varepsilon_{i\alpha})\frac{\partial u_\alpha}{\partial x_j} - \frac{1}{2}\rho_{\varepsilon_{ij}}(\delta_{j\alpha} - 2\varepsilon_{j\alpha})\frac{\partial u_\alpha}{\partial x_i} =$$

$$= \frac{d\rho}{dt} + \frac{1}{2}\rho\delta_{j\alpha}\frac{\partial u_\alpha}{\partial x_j} + \frac{1}{2}\rho\delta_{i\alpha}\frac{\partial u_\alpha}{\partial x_i} = \frac{d\rho}{dt} + \rho\frac{\partial u_k}{\partial x_k} \equiv \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k}$$

It is now clear that the relaxational terms must be introduced in such a way so as not to violate the continuity equation; to this end the relations

$$\rho_{\varepsilon_{ij}}\varphi_{ij} = 0$$

must be satisfied. (Summation here with respect to i and with respect to j is to be understood.)

It is obvious that in the case of small deformations

$$\rho_{\varepsilon_{ij}} = \delta_{ij} + O(\varepsilon_{pq}^2)$$

this relation assumes the form

$$\varphi_{11} + \varphi_{22} + \varphi_{33} = 0$$

It is satisfied automatically for the Kelvin relaxational terms

$$\varphi_{ij} = -\frac{\varepsilon_{ij} - {}^1/_3(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\,\delta_{ij}}{\tau}$$

When considering finite deformations, it is natural to choose the relaxational terms in the following way:

$$\varphi_{ij} = -[\varepsilon_{ij} - \varepsilon_{pq}\rho_{\varepsilon_{pq}}(\rho_{\varepsilon_{11}} + \rho_{\varepsilon_{22}} + \rho_{\varepsilon_{33}})^{-1}\delta_{ij}]\,\tau^{-1} =$$
$$= \frac{1}{2\tau}\left[\delta_{ij} - 2\varepsilon_{ij} + \frac{3\rho}{\rho_{\varepsilon_{11}} + \rho_{\varepsilon_{22}} + \rho_{\varepsilon_{33}}}\delta_{ij}\right]$$

This definition, as may be readily verified, is invariant with respect to the choice of the orthogonal coordinate system and satisfies the condition $\rho_{\varepsilon_{ij}}\varphi_{ij} = 0$. Moreover, τ may be an arbitrary positive function of the parameters defining the state of the material. In particular if $\tau = \infty$ for a tangential stress intensity

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2}$$

less than σ_*, and $\tau = \tau_0$ for greater intensities, we arrive at a viscoelastic medium which possesses, for small τ_0, all the fundamental properties of a plastic material satisfying the von Mises plasticity criterion.

As another example, we present a more general form of the relaxational terms, which in the principal axes of the deformation tensor ($\varepsilon_{ij} = 0\ (i \neq j)$, $\varepsilon_{ii} \equiv \varepsilon_i$), have the form

$$\varphi_i = -\frac{1}{\tau_i}\left[\varepsilon_i - \left(\frac{\varepsilon_1}{\tau_1}\rho_{\varepsilon_1} + \frac{\varepsilon_2}{\tau_2}\rho_{\varepsilon_2} + \frac{\varepsilon_3}{\tau_3}\rho_{\varepsilon_3}\right)\left(\frac{1}{\tau_1}\rho_{\varepsilon_1} + \frac{1}{\tau_2}\rho_{\varepsilon_2} + \frac{1}{\tau_3}\rho_{\varepsilon_3}\right)^{-1}\right]$$

In this case also, it is easy to verify that for arbitrary τ_1, τ_2, τ_3 (we again assume these to be positive) the condition $\rho_{\varepsilon_{ij}}\varphi_{ij} = 0$ is satisfied, which implies the satisfaction of the continuity equation. This form of the relaxational terms makes it possible to model the process of incomplete plasticity, wherein the intensity of the tangential stresses may exceed σ_* on some areas and not on others. Moreover, we can assume that part of the τ_1, τ_2, τ_3 is equal to ∞ or some very large time τ_∞, and a part equal to a small time τ_0. The previous version of the relaxational terms may be obtained from that selected here by putting $\tau_1 = \tau_2 = \tau_3 = \tau$. Putting

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}(\delta_{i\alpha} - 2\varepsilon_{i\alpha})\frac{\partial u_\alpha}{\partial x_j} + \frac{1}{2}(\delta_{j\alpha} - 2\varepsilon_{j\alpha})\frac{\partial u_\alpha}{\partial x_i} + \varphi_{ij}$$

we decompose the deformation rate tensor $d\varepsilon_{ij}/dt$, usually called the total deformation rate tensor, into an elastic part $d\varepsilon_{ij}^e/dt$ and into a plastic part $d\varepsilon_{ij}^p/dt = \varphi_{ij}$. The requirement $\rho\varepsilon_{ij}\,\varphi_{ij} = 0$ represents a statement to the effect that plastic deformations occur without change of volume (see, for example, [7]).

In calculating elastic-plastic flows, Wilkins [8] managed to satisfy the von Mises criterion through a rather arbitrary process of normalization of the deviator. Such a normalization leaves unclarified the matter of how to verify that the laws of thermodynamics are satisfied. Apparently, the introduction of a dependence of the relaxation time on the tangential stress intensity may lead automatically to almost an exact satisfaction of the plasticity conditions. Validity of the law of entropy increase for a given way of writing the relaxational terms will be verified in Sec. 3.

We make several remarks concerning the compatibility conditions for the model in question. It is well known that we can associate with each metric tensor $g_{ik} = \delta_{ik} - 2\varepsilon_{ik}$ the Riemann–Christoffel curvature tensor

$$R_{ik\lambda\mu} = \frac{1}{2}\left\{\frac{\partial^2 g_{i\mu}}{\partial x^k\,\partial x^\lambda} - \frac{\partial^2 g_{i\lambda}}{\partial x^k\,\partial x^\mu} - \frac{\partial^2 g_{k\mu}}{\partial x^i\,\partial x^\lambda} + \frac{\partial^2 g_{k\lambda}}{\partial x^i\,\partial x^\mu}\right\} - g^{\rho\sigma}\Gamma_{\sigma,\,\lambda i}\Gamma_{\rho,\,\mu k} + g^{\rho\sigma}\Gamma_{\sigma,\,\lambda k}\Gamma_{\rho,\,\mu i}$$

Here $\Gamma_{r,\,ik}$ are the Christoffel symbols

$$\Gamma_{r,\,ik} = \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^r}\right)$$

A nonzero value of $R_{ik\lambda\mu}$ characterizes the "incompatibility" of the metric deformation tensor with Euclidean three-dimensional space. In three-dimensional space the tensor $R_{ik\lambda\mu}$ has only six different nonzero components; thus, instead of the four-valent curvature tensor $R_{ik\lambda\mu}$, we can consider the two-valent tensor $R_{i\mu}$ with the same nonzero components

$$R_{i\mu} = g^{k\lambda}R_{ik\lambda\mu}$$

or the Einstein tensor

$$G_{ik} = R_{ik} - {}^1/_2\, g_{ik}R \qquad (R = g^{\lambda\mu}R_{\lambda\mu})$$

(see, for example [9, 10]).

The reader interested in the relationship between the curvature tensor and the density of dislocations should consult §§ 9, 12, 14 of the supplement to [11], and also [7]. These papers are devoted to the linear theory of elasticity. The role of the curvature tensor as the incompatibility tensor was considered in [4].

We consider next how the tensor G_{ik} varies with the time when the variation of the deformation tensor is described by the equations

$$\frac{d\varepsilon_{ik}}{dt} = \frac{1}{2}(\delta_{i\alpha} - 2\varepsilon_{i\alpha})\frac{\partial u_\alpha}{\partial x_k} + \frac{1}{2}(\delta_{k\alpha} - 2\varepsilon_{k\alpha})\frac{\partial u_\alpha}{\partial x_i} + \varphi_{ik}$$

If we set the relaxational terms $\varphi_{ik} = 0$, it is then easy to see that the motion described by these equations can be regarded as a continuous variation of the coordinates in the initial space with the metric g_{ik}°. In addition, all the tensors transform according to the same rule. In particular

$$\frac{dg_{ik}}{dt} + g_{i\alpha}\frac{\partial u_\alpha}{\partial x_k} + g_{k\alpha}\frac{\partial u_\alpha}{\partial x_i} = 0$$

Therefore for G_{ik} it is necessary to write

$$\frac{dG_{ik}}{dt} + G_{i\alpha}\frac{\partial u_\alpha}{\partial x_k} + G_{k\alpha}\frac{\partial u_\alpha}{\partial x_i} = 0$$

Finally, this equation, being a relationship along the characteristics (streamlines) of the elasticity theory equations formulated above, can be obtained from these equations through differentiation and taking corresponding linear combinations. For the case in which the relaxational terms φ_{ik} are nonzero, the equations for the G_{ik} will have right-hand sides* Φ_{ik}, where

$$\frac{dG_{ik}}{dt} + G_{i\alpha}\frac{\partial u_\alpha}{\partial x_k} + G_{k\alpha}\frac{\partial u_\alpha}{\partial x_i} = \Phi_{ik}$$

$$\begin{aligned}\Phi_{ik} = {} & \left(\frac{1}{2}g_{ik}g^{\varkappa\lambda}g^{\alpha\beta} - \delta_\varkappa{}^i\delta_\lambda{}^k g^{\alpha\beta}\right)\left\{\frac{\partial^2\varphi_{\alpha\beta}}{\partial x^\varkappa\,\partial x^\lambda} - \frac{\partial^2\varphi_{\alpha\lambda}}{\partial x^\varkappa\,\partial x^\beta} - \frac{\partial^2\varphi_{\varkappa\beta}}{\partial x^\alpha\,\partial x^\lambda} + \frac{\partial^2\varphi_{\varkappa\lambda}}{\partial x^\alpha\,\partial x^\beta}\right\} + g^{\varkappa\mu}\Gamma^\alpha_{k\varkappa}\Pi_{\alpha,\,\mu i} + g^{\varkappa\mu}\Gamma^\alpha_{\mu i}\Pi_{\alpha,\,k\varkappa} - \\ & - g^{\varkappa\mu}\Gamma^\alpha_{ki}\Pi_{\alpha,\,\mu\varkappa} - g^{\varkappa\mu}\Gamma^\alpha_{\mu\varkappa}\Pi_{\alpha,\,ki} - {}^1/_2\, g_{ik}g^{\varkappa\lambda}g^{\beta\mu}\Gamma^\alpha_{\mu\varkappa}\Pi_{\alpha,\,\lambda\beta} + \\ & + {}^1/_2\, g_{ik}g^{\varkappa\lambda}g^{\beta\mu}\Gamma^\alpha_{\lambda\varkappa}\Pi_{\alpha,\,\mu\beta} - {}^1/_2\, g_{ik}g^{\varkappa\lambda}g^{\beta\mu}\Gamma^\alpha_{\lambda\beta}\Pi_{\alpha,\,\mu\varkappa} + {}^1/_2\, g_{ik}g^{\varkappa\lambda}g^{\beta\mu}\Gamma^\alpha_{\mu\beta}\Pi_{\alpha,\,\lambda\varkappa} + \\ & + 2g^{\varkappa\mu}(\Gamma_{\alpha,\,ki}\Gamma_{\beta,\,\mu\varkappa} - \Gamma_{\alpha,\,k\varkappa}\Gamma_{\beta,\,\mu i})\varphi^{\alpha\beta} - g_{ik}g^{\varkappa\lambda}g^{\gamma\mu} \times \\ & \times (\Gamma_{\alpha,\,\lambda\varkappa}\Gamma_{\beta,\,\mu\gamma} - \Gamma_{\alpha,\,\lambda\gamma}\Gamma_{\beta,\,\mu\varkappa})\varphi^{\alpha\beta} + 2\varphi^{\varkappa\mu}R_{\varkappa ik\mu} - 2g_{ik}\varphi^{\varkappa\mu}R_{\varkappa\mu} + R\varphi_{ik}\end{aligned}$$

$$\Pi_{r,\,\alpha\beta} = \frac{\partial\varphi_{r\alpha}}{\partial x^\beta} + \frac{\partial\varphi_{r\beta}}{\partial x^\alpha} - \frac{\partial\varphi_{\alpha\beta}}{\partial x^r}$$

*In the equations for the Φ_{ik} appearing below, we have used the contravariant components g^{ik} of the metric tensor and the Christoffel symbols $\Gamma^\alpha_{\mu\varkappa}$ corresponding to them.

In the theory of an elastic crystalline medium the nonzero curvature tensor $R_{ik\lambda\mu}$, and the tensor G_{ik} associated with it, essentially characterizes the nonzero density of dislocations (see [7]). The terms Φ_{ik}, appearing in the right-hand sides of the equations for the G_{ik}, can be treated as a characteristic of the dislocation "source" density in the model of the viscoelastic medium considered. A study of the curvature tensor lies somewhat outside the scope of this study. We therefore limit ourselves to the brief remarks made here.

We remark also that the divergence of the contravariant components of the tensor G^{ik} is equal to zero,

$$\nabla_k G^{ik} = \frac{\partial}{\partial x^k} G^{ik} + \Gamma^i_{\alpha k} G^{\alpha k} + \Gamma^k_{\alpha k} G^{i\alpha} = 0$$

(see [10], p. 624).

3. The Complete System of Equations, Their Transformation, and Thermodynamic Identities. The concepts formulated above lead to a system of differential equations describing the behavior of a continuous medium with time. This system consists of ten equations in ten unknowns

$$\varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{33},\ \varepsilon_{12},\ \varepsilon_{13},\ \varepsilon_{23},\ S,\ u_1,\ u_2,\ u_3$$

$$\frac{\partial \rho (E + {}^1/_2\, u_i u_i)}{\partial t} + \frac{\partial\, [\rho u_k (E + {}^1/_2\, u_i u_i) - u_i \sigma_{ik}]}{\partial x_k} = 0 \qquad (3.1)$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial (\rho u_i u_k - \sigma_{ik})}{\partial x_k} = 0 \qquad (i = 1, 2, 3)$$

$$\frac{\partial \varepsilon_{ii}}{\partial t} + u_k \frac{\partial \varepsilon_{ii}}{\partial x_k} - \frac{\partial u_i}{\partial x_i} + 2\varepsilon_{i\alpha} \frac{\partial u_\alpha}{\partial x_i} = \varphi_{ii} \qquad (i = 1, 2, 3)$$

$$\frac{\partial 2\varepsilon_{ij}}{\partial t} + u_k \frac{\partial 2\varepsilon_{ij}}{\partial x_k} - \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + 2\varepsilon_{i\alpha} \frac{\partial u_\alpha}{\partial x_j} + 2\varepsilon_{j\alpha} \frac{\partial u_\alpha}{\partial x_i} = 2\varphi_{ij}$$
$$(ij) = (12),\ (13),\ (23)$$

In this system we have not included the continuity equation and the equation for the conservation (increase) of entropy, since these equations are consequences of this system. Derivation of the continuity equation was given in Sec. 2. Our next concern is the law for the growth of entropy. We remark also that in the sequel (see Sec. 4), instead of the laws for the conservation of momentum

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial (\rho u_i u_k - \sigma_{ik})}{\partial x_k} = 0$$

it is necessary to include in the system the nondivergent Euler equations for the velocities

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial \sigma_{ik}}{\partial x_k} = 0$$

which follow from the laws for the conservation of momentum and the continuity equation

$$\rho \left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \right) - \frac{\partial \sigma_{ik}}{\partial x_k} = \frac{\partial \rho u_i}{\partial t} + \frac{\partial (\rho u_i u_k - \sigma_{ik})}{\partial x_k} - u_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} \right) = 0$$

and also, in place of the law for the conservation of entropy,

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} = 0$$

we use the equation

$$\frac{dS}{dt} \equiv \frac{\partial S}{\partial t} + u_k \frac{\partial S}{\partial x_k} = 0$$

which is obtained from the preceding equation and the continuity equation

$$\rho \frac{dS}{dt} = \left(\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u_k}{\partial x_k} \right) - S \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} \right) = 0$$

We can write the system of Eqs. (3.1) in the form

$$\frac{\partial L_{q_i}}{\partial t}+\frac{\partial M^k_{q_i}}{\partial x_k}+\frac{\partial H^k_{q_i}}{\partial x_k}+A^n\frac{\partial h^{kn}_{q_i}}{\partial x_k}+h^{kn}_{q_i}\frac{\partial A^n}{\partial x_k}=0 \qquad (i=1, 2, \ldots, i_0) \tag{3.2}$$

$$\frac{\partial L_{r_j}}{\partial t}+\frac{\partial M^k_{r_j}}{\partial x_k}+\frac{\partial H^k_{r_j}}{\partial x_k}+A^n\frac{\partial h^{kn}_{r_j}}{\partial x_k}=f_j \qquad (j=1, 2, \ldots, j_0)$$

where L, M^k, H^k, h^{kn} are arbitrary functions, and A^n and f_j are arbitrary functions of $q_1, \ldots, q_{i_0}, r_1, \ldots, r_{j_0}$, where H^k and h^{kn} are homogeneous functions. The nature of the homogeneity must be such as to imply the Euler identities:

$$q_iH^k_{q_i}+r_jH^k_{r_j}=H^k, \quad q_ih^{kn}_{q_i}=0, \quad r_jh^{kn}_{r_j}=h^{kn} \tag{3.3}$$

The choice of the variables q_i and r_j and the generating functions will be described later; for the present we show that if the first i_0 equations of the system (3.2) are multiplied by the corresponding q_i, the remaining j_0 equations by the corresponding r_j, and if the results are added together, then, using the homogeneity equations (3.3), we obtain the conservation law

$$\frac{\partial(q_iL_{q_i}+r_jL_{r_j}-L)}{\partial t}+\frac{\partial(q_iM^k_{q_i}+r_jM^k_{r_j}-M^k)}{\partial x_k}=r_jf_j$$

The variables q_i and r_j, the generating functions L, M^k, H^k, h^{kn}, and the functions A^n for the system (3.1) have the form

$$q_0=1/E_S, \quad q_i=-u_i/E_S \quad (i=1, 2, 3)$$

$$r_1=-\frac{1}{E_S}\left[\left(E-SE_S-\frac{u_iu_i}{2}\right)\rho_{\varepsilon_{11}}+\rho E_{\varepsilon_{11}}\right]$$

$$r_2=-\frac{1}{E_S}\left[\left(E-SE_S-\frac{u_iu_i}{2}\right)\rho_{\varepsilon_{22}}+\rho E_{\varepsilon_{22}}\right]$$

$$r_3=-\frac{1}{E_S}\left[\left(E-SE_S-\frac{u_iu_i}{2}\right)\rho_{\varepsilon_{33}}+\rho E_{\varepsilon_{33}}\right]$$

$$r_4=-\frac{1}{E_S}\left[\left(E-SE_S-\frac{u_iu_i}{2}\right)\rho_{\varepsilon_{12}}+\rho E_{\varepsilon_{12}}\right]$$

$$r_5=-\frac{1}{E_S}\left[\left(E-SE_S-\frac{u_iu_i}{2}\right)\rho_{\varepsilon_{13}}+\rho E_{\varepsilon_{13}}\right]$$

$$r_6=-\frac{1}{E_S}\left[\left(E-SE_S-\frac{u_iu_i}{2}\right)\rho_{\varepsilon_{23}}+\rho E_{\varepsilon_{23}}\right]$$

$$L=-\frac{1}{2E_S}\left(E-SE_S-\frac{u_iu_i}{2}\right)(\rho+\rho_{\varepsilon_{11}}+\rho_{\varepsilon_{22}}+\rho_{\varepsilon_{33}})-\frac{\rho}{E_S}\varepsilon_{ik}E_{\varepsilon_{ik}}$$

$$M^k=u_kL=-q_kq_0^{-1}L$$

$$H^1=\frac{q_1r_1+q_2r_4+q_3r_5}{q_0}, \quad H^2=\frac{q_1r_4+q_2r_5+q_3r_6}{q_0}, \quad H^3=\frac{q_1r_5+q_2r_6+q_3r_3}{q_0}$$

$$h^{11}=-\frac{q_1r_1}{q_0}, \quad h^{12}=\frac{q_1r_2-2q_2r_4}{q_0}, \quad h^{13}=\frac{q_1r_3-2q_3r_5}{q_0}, \quad h^{14}=-\frac{q_2r_1}{q_0}$$

$$h^{15}=-\frac{q_3r_1}{q_0}, \quad h^{16}=\frac{q_1r_6-q_2r_5-q_3r_4}{q_0}$$

$$h^{21}=\frac{q_2r_1-2q_1r_4}{q_0}, \quad h^{22}=-\frac{q_2r_2}{q_0}, \quad h^{23}=\frac{q_2r_3-2q_3r_6}{q_0}, \quad h^{24}=-\frac{q_1r_2}{q_0}$$

$$h^{25}=\frac{q_2r_5-q_1r_6-q_3r_4}{q_0}, \quad h^{26}=-\frac{q_3r_2}{q_0}$$

$$h^{31}=\frac{q_3r_1-2q_1r_5}{q_0}, \quad h^{32}=\frac{q_3r_2-2q_2r_6}{q_0}, \quad h^{33}=-\frac{q_3r_3}{q_0}, \quad h^{34}=\frac{q_3r_4-q_1r_6-q_2r_5}{q_0}$$

$$h^{35}=-\frac{q_1r_3}{q_0}, \quad h^{36}=-\frac{q_2r_3}{q_0}$$

$$A^n=L_{r_n}, \text{ i.e., } A^1=\varepsilon_{11}, \ A^2=\varepsilon_{22}, \ A^3=\varepsilon_{33}, \ A^4=2\varepsilon_{12}, \ A^5=2\varepsilon_{13}, \ A^6=2\varepsilon_{23}$$

$$f_1=\varphi_{11}, \ f_2=\varphi_{22}, \ f_3=\varphi_{33}, \ f_4=2\varphi_{12}, \ f_5=2\varphi_{13}, \ f_6=2\varphi_{23}$$

(the form of the φ_{ij} was considered in Section 2).

The conservation law can then be written as follows:

$$\frac{\partial\rho S}{\partial t}+\frac{\partial\rho Su_k}{\partial x_k}=\frac{\rho}{E_S}Q$$

$$Q=-[E_{\varepsilon_{ij}}\varphi_{ij}+\rho^{-1}(E-SE_S-{}^1/_2\,u_ku_k)\rho_{\varepsilon_{ij}}\varphi_{ij}]=-E_{\varepsilon_{ij}}\varphi_{ij}$$

Here we have used the equation $\rho_{\varepsilon_{ij}} \varphi_{ij} = 0$, which assures satisfaction of the continuity equation (see Sec. 2).

Thus the form of the equations shown is suitable for the study of thermodynamic relationships, and the variables q_i, r_j represent "integrating factors" by which it is necessary to multiply the equations of the system to obtain the law for the conservation (or growth, if $Q > 0$) of entropy. Analogous and substantially simpler forms of the systems of equations of mathematical physics were, in fact, considered in [12-15]. In these papers it was shown that the hyperbolicity of the systems in the Friedrichs sense is a consequence of the canonical forms used for writing the equations.

The reduction of the equations of nonlinear elasticity theory to a "thermodynamic" form of this type was undertaken initially with the same goal in mind, namely, to prove the hyperbolicity of the system and to obtain estimates of the energy integrals for the derived solutions. However, after such a reduction was carried out, it became clear that in this case the symmetric hyperbolicity of the system is not an automatic consequence of the form (3.2). The point is that because of the diverse character of the homogeneity of the functions h^{kn} in the variables q_i and r_j, the conservation law for the system holds but the matrices

$$R^k = \begin{pmatrix} (M^k_{q_i q_m} + H^k_{q_i q_m} + A^n h^{kn}_{q_i q_m} + h^{kn}_{q_i} A^n_{q_m})(M^k_{q_i r_j} + H^k_{q_i r_j} + A^n h^{kn}_{q_i r_j} + h^{kn} A^n_{r_j}) \\ (M^k_{r_j q_i} + H^k_{r_j q_i} + A^n h^{kn}_{r_j q_i})(M^k_{r_j r_l} + H^k_{r_j r_l} + A^n h^{kn}_{r_j r_l}) \end{pmatrix}$$

in the quasilinear way of writing the system (3.2), namely,

$$\begin{pmatrix} L_{q_i q_m} & L_{q_i r_j} \\ L_{r_j q_i} & L_{r_j r_l} \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} q \\ r \end{pmatrix} + R^k \frac{\partial}{\partial x_k}\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

are not symmetric.

This result has lead to the necessity of using an altogether different method for calculation of the characteristics and a reduction of the system to a symmetric hyperbolic form, namely, one based on broadening the initial system by including equations obtained by differentiating the Euler equations for the velocities. We describe such a symmetrization in Sec. 4; for the present we study the conditions the equation of state $E(a_1, a_2, a_3, S)$ must satisfy in order that the relaxation of the tangential stresses, described by the form adopted for the Kelvin terms φ_{ij}, will lead to the condition $Q > 0$, i.e., to the law for the growth of entropy.

In a system of coordinates whose axes are directed along the principal axes of the deformation tensor, we have

$$E_{\varepsilon_{ii}} \equiv E_{\varepsilon_i} = a_i^3 E_{a_i}, \qquad E_{\varepsilon_{ij}} = 0 \qquad (i \neq j)$$

Therefore, in this system of coordinates

$$Q = -E_{\varepsilon_{ij}} \varphi_{ij} = \frac{1}{\tau} E_{\varepsilon_i}\left[\varepsilon_i - \left(\frac{\varepsilon_1}{\tau_1}\rho_{\varepsilon_1} + \frac{\varepsilon_2}{\tau_2}\rho_{\varepsilon_2} + \frac{\varepsilon_3}{\tau_3}\rho_{\varepsilon_3}\right) \times \right.$$
$$\left. \times \left(\frac{1}{\tau_1}\rho_{\varepsilon_1} + \frac{1}{\tau_2}\rho_{\varepsilon_2} + \frac{1}{\tau_3}\rho_{\varepsilon_3}\right)^{-1}\right] = \frac{1}{2}\left[\frac{1}{\tau_1\tau_2}(a_1^2 - a_2^2)^2 \frac{a_1 E_{a_1} - a_2 E_{a_2}}{a_1^2 - a_2^2} + \right.$$
$$\left. + \frac{1}{\tau_2\tau_3}(a_2^2 - a_3^2)^2 \frac{a_2 E_{a_2} - a_3 E_{a_3}}{a_2^2 - a_3^2} + \frac{1}{\tau_3\tau_1}(a_3^2 - a_1^2)^2 \frac{a_3 E_{a_3} - a_1 E_{a_1}}{a_3^2 - a_1^2}\right]\left(\frac{a_1^2}{\tau_1} + \frac{a_2^2}{\tau_2} + \frac{a_3^2}{\tau_3}\right)^{-1}$$

For Q to be positive it is sufficient to have the following inequalities satisfied:

$$\frac{a_1 E_{a_1} - a_2 E_{a_2}}{a_1^2 - a_2^2} > 0, \qquad \frac{a_2 E_{a_2} - a_3 E_{a_3}}{a_2^2 - a_3^2} > 0, \qquad \frac{a_3 E_{a_3} - a_1 E_{a_1}}{a_3^2 - a_1^2} > 0$$

In Sec. 5 it is shown that these inequalities are a necessary consequence of the hyperbolicity of the system in question.

4. Reduction of the System to a Form Containing Second Derivatives of the Velocities; Determination of Conditions for the Characteristics to Be Real. As a starting point for further study we take the system described earlier

$$\rho \frac{du_k}{dt} - \frac{\partial \sigma_{ki}}{\partial \varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_i} - \frac{\partial \sigma_{ki}}{\partial S} \frac{\partial S}{\partial x_i} = f_k \tag{4.1}$$

$$\frac{d\varepsilon_{mn}}{dt} = \frac{1}{2}\left[(\delta_{ml} - 2\varepsilon_{ml})\delta_{jn} + (\delta_{nl} - 2\varepsilon_{nl})\delta_{jm}\right]\frac{\partial u_l}{\partial x_j} + \varphi_{mn} \tag{4.2}$$

$$\frac{dS}{dt} = \varkappa = \frac{1}{E_S}Q \tag{4.3}$$

We apply the operator d/dt to each of the Eqs. (4.1) and into the result we substitute the derivatives with respect to $\partial/\partial x_i$, obtained from Eqs. (4.2) and (4.3). We carry out this reduction in detail.

Noting that

$$\frac{\partial}{\partial x_i}\frac{d}{dt} = \frac{\partial}{\partial t}\frac{\partial}{\partial x_i} + u_\alpha\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial}{\partial x_\alpha} = \frac{d}{dt}\frac{\partial}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial}{\partial x_\alpha} \tag{4.4}$$

we differentiate Eqs. (4.2) and (4.3) with respect to x_i:

$$\begin{gathered}\frac{\partial}{\partial x_i}\frac{d\varepsilon_{mn}}{dt} = \frac{1}{2}\left[(\delta_{ml} - 2\varepsilon_{ml})\delta_{jn} + (\delta_{nl} - 2\varepsilon_{nl})\delta_{jm}\right]\frac{\partial^2 u_l}{\partial x_i\partial x_j} - \frac{\partial u_l}{\partial x_j}\left(\delta_{jn}\frac{\partial\varepsilon_{ml}}{\partial x_i} + \delta_{jm}\frac{\partial\varepsilon_{nl}}{\partial x_i}\right) + \frac{\partial\varphi_{mn}}{\partial x_i}\\ \frac{\partial}{\partial x_i}\frac{dS}{dt} = \frac{\partial\varkappa}{\partial x_i}\end{gathered} \tag{4.5}$$

To Eq. (4.1) we apply the operator d/dt, using the commutation rule for the operators d/dt and $\partial/\partial x_i$

$$\begin{gathered}\rho\left(\frac{d}{dt}\right)^2 u_k - \frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\frac{\partial}{\partial x_i}\frac{d\varepsilon_{mn}}{dt} - \frac{\partial\sigma_{ki}}{\partial S}\frac{\partial}{\partial x_i}\frac{dS}{dt} +\\ + \frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial\varepsilon_{mn}}{\partial x_\alpha} + \frac{\partial\sigma_{ki}}{\partial S}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial S}{\partial x_\alpha} + \frac{du_k}{dt}\frac{d\rho}{dt} -\\ - \frac{d}{dt}\left(\frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\right)\frac{\partial\varepsilon_{mn}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial\sigma_{ki}}{\partial S}\right)\frac{\partial S}{\partial x_i} = \frac{df_k}{dt}\end{gathered} \tag{4.6}$$

and we replace the second and third terms by their expressions (4.5). To simplify the writing we also introduce the notation a^{ij}_{kl} for the following tensor:

$$\begin{gathered}a^{ij}_{kl} = \frac{1}{2}\frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\left[(\delta_{ml} - 2\varepsilon_{ml})\delta_{jn} + (\delta_{nl} - 2\varepsilon_{nl})\delta_{jm}\right] =\\ = \frac{1}{2}(\delta_{ml} - 2\varepsilon_{ml})\frac{\partial\sigma_{ki}}{\partial\varepsilon_{mj}} + \frac{1}{2}(\delta_{nl} - 2\varepsilon_{nl})\frac{\partial\sigma_{ki}}{\partial\varepsilon_{jn}} =\\ = {}^1/_2\,\rho\,(\delta_{k\alpha} - 2\varepsilon_{k\alpha})(\delta_{\beta l} - 2\varepsilon_{\beta l})(E_{\varepsilon_{\alpha i}\varepsilon_{\beta j}} + E_{\varepsilon_{\alpha i}\varepsilon_{j\beta}}) -\\ - \rho\delta_{lj}(\delta_{k\alpha} - 2\varepsilon_{k\alpha})E_{\varepsilon_{\alpha i}} - \rho\delta_{kj}(\delta_{\alpha l} - 2\varepsilon_{\alpha l})E_{\varepsilon_{\alpha i}} - \rho(\delta_{kl} - 2\varepsilon_{kl})E_{\varepsilon_{ji}}\end{gathered} \tag{4.7}$$

Using an orthogonal system of coordinates, we do not distinguish between covariant and contravariant tensor components. The indices ij in the tensor a^{ij}_{kl} are raised merely to distinguish their role from that of the indices kl in the successive formulas and to make these formulas easy to read. We note now that

$$\frac{1}{\rho}a^{ij}_{kl}\frac{\partial^2 u_l}{\partial x_i\partial x_j} + \frac{1}{\rho}a^{ji}_{kl}\frac{\partial^2 u_l}{\partial x_j\partial x_i} = \frac{1}{2\rho}(a^{ij}_{kl} + a^{ji}_{kl})\frac{\partial^2 u_l}{\partial x_i\partial x_j} = A^{ij}_{kl}\frac{\partial^2 u_l}{\partial x_i\partial x_j}$$

We denote by A^{ij}_{kl} the tensor symmetrized with respect to i, j

$$A^{ij}_{kl} = (a^{ij}_{kl} + a^{ji}_{kl})/2\rho \tag{4.8}$$

As a result, the Eq. (4.6) may be rewritten as

$$\begin{gathered}\left(\frac{d}{dt}\right)^2 u_k - A^{ij}_{kl}\frac{\partial^2 u_l}{\partial x_i\partial x_j} + \frac{1}{\rho}\frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\left[\frac{\partial u_l}{\partial x_j}\left(\delta_{jn}\frac{\partial\varepsilon_{ml}}{\partial x_i} + \delta_{jm}\frac{\partial\varepsilon_{nl}}{\partial x_i}\right) - \frac{\partial\varphi_{mn}}{\partial x_i}\right] -\\ - \frac{1}{\rho}\frac{\partial\sigma_{ki}}{\partial S}\frac{\partial\varkappa}{\partial x_i} + \frac{1}{\rho}\frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial\varepsilon_{mn}}{\partial x_\alpha} + \frac{1}{\rho}\frac{\partial\sigma_{ki}}{\partial S}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial S}{\partial x_\alpha} -\\ - \frac{1}{\rho}\frac{d}{dt}\left(\frac{\partial\sigma_{ki}}{\partial\varepsilon_{mn}}\right)\frac{\partial\varepsilon_{mn}}{\partial x_i} - \frac{1}{\rho}\frac{d}{dt}\left(\frac{\partial\sigma_{ki}}{\partial S}\right)\frac{\partial S}{\partial x_i} = \frac{1}{\rho}\frac{df_k}{dt} + \delta_{ij}\frac{\partial u_i}{\partial x_j}\frac{du_k}{dt}\end{gathered} \tag{4.9}$$

the last term on the right side having been obtained with the help of the continuity equation

$$\delta_{ij}\frac{\partial u_i}{\partial x_j}\frac{du_k}{dt} = -\frac{1}{\rho}\frac{d\rho}{dt}\frac{du_k}{dt}$$

We now express the coefficients $d/dt\,(\partial\sigma_{ki}/\partial\varepsilon_{mn})$ and $d/dt\,(\partial\sigma_{ki}/\partial S)$ in terms of $\varepsilon_{\alpha\beta}$ and $\partial u_l/\partial x_j$ with the aid of Eqs. (4.2) and (4.3).

If in the system (4.1)-(4.3) under study we replace the first order Eq. (4.1) for u_k by the resulting second order Eq. (4.9), we may reduce the latter to the following form:

$$\left(\frac{d}{dt}\right)^2 u - A^{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} - B^i\frac{\partial v}{\partial x_i} = F \qquad (4.10)$$
$$\frac{dv}{dt} = C^k\frac{\partial u}{\partial x_k} + \varphi$$

Here

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ S \end{pmatrix}$$

A^{ij} are the third-order square matrices 3×3

$$A^{ij} = A^{ij}(v) = \begin{pmatrix} A^{ij}_{11} & A^{ij}_{12} & A^{ij}_{13} \\ A^{ij}_{21} & A^{ij}_{22} & A^{ij}_{23} \\ A^{ij}_{31} & A^{ij}_{32} & A^{ij}_{33} \end{pmatrix}$$

B^k are rectangular matrices of order 3×7

$$B^k = B^k(v, \partial u/\partial x)$$

C^k are rectangular matrices of order 7×3

$$C^k = C^k(v)$$

On the right side of the system there appear the vectors F and φ, of dimensions 3 and 7, respectively,

$$F = F(v, du/dt, \partial u/\partial x), \qquad \varphi = \varphi(v)$$

We obtain from Eqs. (4.7) and (4.8) an equation expressing the matrix elements A^{ij}_{kl} in terms of derivatives of E with respect to the deformation tensor components

$$\begin{aligned} A_{kl}{}^{ij} = {}^1/_4(\delta_{k\alpha} - 2\varepsilon_{k\alpha})(\delta_{\beta l} - 2\varepsilon_{\beta l})(E_{\varepsilon_{\alpha i}\varepsilon_{\beta j}} + E_{\varepsilon_{\alpha i}\varepsilon_{j\beta}} + E_{\varepsilon_{\alpha j}\varepsilon_{\beta i}} + E_{\varepsilon_{\alpha j}\varepsilon_{i\beta}}) - \\ - {}^1/_2(\delta_{lj}E_{\varepsilon_{\alpha i}} + \delta_{li}E_{\varepsilon_{\alpha j}})(\delta_{k\alpha} - 2\varepsilon_{k\alpha}) - {}^1/_2(\delta_{kj}E_{\varepsilon_{\alpha i}} + \delta_{ki}E_{\varepsilon_{\alpha j}})(\delta_{l\alpha} - 2\varepsilon_{l\alpha}) - \\ - {}^1/_2(E_{\varepsilon_{ij}} + E_{\varepsilon_{ji}})(\delta_{kl} - 2\varepsilon_{kl}) \end{aligned} \qquad (4.11)$$

From this equation there follow the symmetry relations

$$A_{kl}{}^{ij} = A_{lk}{}^{ji} = A_{kl}{}^{ji}$$

In calculating the characteristics of the system (4.10) it is necessary to keep in mind that in the characteristic equation only coefficients of the highest derivatives must appear, i.e., coefficients of the second derivatives of the vector u and of the first derivatives of the vector v. Let I_p be unit matrix of order $p\times p$ and let Ω denote the expression $\omega + u_i\xi_i$ (ω, ξ_i are the components of the wave vector or of the vector normal to the characteristic surface). The equation of the characteristics has the form

$$\begin{vmatrix} \Omega^2 I_3 - \xi_i\xi_j A^{ij} - \xi_i B^i \\ 0 \quad \cdot \quad \Omega I_7 \end{vmatrix} = (\omega + u_i\xi_i)^7 \det\|(\omega + u_i\xi_i)^2 I_3 - A^{ij}\xi_i\xi_j\| = 0$$

The factor $(\omega + u_i\xi_i)^7$ in the characteristic equation shows that the streamline is a multiple characteristic. For arbitrary fixed ξ_k the sixth degree equation

$$\det\|A^{ij}\xi_i\xi_j - (\omega + u_i\xi_i)^2 I_3\| = 0$$

has, on account of the symmetry of the matrices A^{ij}, the real roots $\xi_k\ (\omega + u_i\xi_i)^2$.

It is evident from this that for the system to be hyperbolic, i.e., for the roots ω to be real, it is necessary that for all ξ_k ($\xi_1^2+\xi_2^2+\xi_3^2 \neq 0$) the third order matrix $A^{ij}\xi_i\xi_j$ be positive definite.

In the sequel we supply various equations for calculating the elements A^{ij}_{kl} in a system of coordinates connected with the principal axes of the deformation tensor (Sec. 5) and we show how the system (4.10) can be reduced to a symmetric hyperbolic Friedrichs-type system of equations of the first order (Sec. 6).

5. Calculation of the Elements of the Matrix A^{ij}_{kl} in the Principal Axes of the Deformation Tensor.

We show that in the principal axes of the deformation tensor the matrix $A = \|A_{ij}\|$ has the following form:

$$\|A^{ij}\| = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix} = \left(\begin{array}{ccc|ccc|ccc} L_1 & 0 & 0 & 0 & {}^1/{}_2N_3 & 0 & 0 & 0 & {}^1/{}_2N_2 \\ 0 & e^{2d_1}M_3 & 0 & {}^1/{}_2N_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2d_1}M_2 & 0 & 0 & 0 & {}^1/{}_2N_2 & 0 & 0 \\ \hline 0 & {}^1/{}_2N_3 & 0 & e^{2d_2}M_3 & 0 & 0 & 0 & 0 & 0 \\ {}^1/{}_2N_3 & 0 & 0 & 0 & L_2 & 0 & 0 & 0 & {}^1/{}_2N_1 \\ 0 & 0 & 0 & 0 & 0 & e^{2d_2}M_1 & 0 & {}^1/{}_2N_1 & 0 \\ \hline 0 & 0 & {}^1/{}_2N_2 & 0 & 0 & 0 & e^{2d_3}M_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {}^1/{}_2N_1 & 0 & e^{2d_3}M_1 & 0 \\ {}^1/{}_2N_2 & 0 & 0 & 0 & {}^1/{}_2N_1 & 0 & 0 & 0 & L_3 \end{array}\right)$$

where

$$L_1 = a_1^2 E_{a_1a_1}, \quad M_1 = (a_1a_2a_3)^{2/3}\frac{a_2E_{a_2} - a_3E_{a_3}}{a_2^2 - a_3^2}$$
$$N_1 = a_2a_3\left[E_{a_2a_3} - \frac{a_2E_{a_3} - a_3E_{a_2}}{a_2^2 - a_3^2}\right] \tag{5.1}$$

Equations for the remaining L_i, M_i, N_i are obtained from Eq. (5.1) through a cyclic interchange of subscripts. In terms of the parameters ρ, D, Δ, d_i the coefficients L_1, M_1, N_1 are expressed as follows:

$$\begin{aligned} L_1 = {} & 2\rho E_\rho + \rho^2 E_{\rho\rho} + ({}^2/{}_3 - d_1)E_D + {}^1/{}_3(2d_1 - D - 3d_2d_3)E_\Delta + \\ & + d_1^2 E_{DD} + (\Delta/d_1 + D/3)^2 E_{\Delta\Delta} - 2\rho d_1 E_{\rho D} - \\ & - 2\rho(\Delta/d_1 + D/3)E_{\rho\Delta} + 2(\Delta + d_1D/3)E_{D\Delta} \\ M_1 = {} & \frac{2(d_2 - d_3)}{e^{2d_2} - e^{2d_3}}\,\frac{E_D - d_1E_\Delta}{2} \\ N_1 = {} & 2\rho E_\rho + \rho^2 E_{\rho\rho} - {}^1/{}_3E_D + {}^1/{}_3(2d_1 - D)E_\Delta + \Delta d_1^{-1}E_{DD} + \\ & + \left(d_1\Delta - \frac{D\Delta}{9d_1} - \frac{2d_1^2}{9}D\right)E_{\Delta\Delta} + d_1\rho E_{\rho D} + \rho\left(\frac{\Delta}{d_1} + \frac{D}{3}\right)E_{\rho\Delta} + \\ & + \left(\frac{2}{3}d_1D - \Delta\right)E_{D\Delta} - \frac{d_2e^{2d_3} - d_3e^{2d_2}}{e^{2d_3} - e^{2d_2}}E_D - \frac{d_3e^{2d_3} - d_2e^{2d_2}}{e^{2d_3} - e^{2d_2}}d_1E_\Delta \end{aligned} \tag{5.2}$$

In the principal axes the characteristic equation assumes the form

$$0 = \begin{vmatrix} L_1\xi_1^2 + e^{2d_2}M_3\xi_2^2 + e^{2d_3}M_2\xi_3^2 - \Omega^2 & N_3\xi_1\xi_2 & N_2\xi_1\xi_3 \\ N_3\xi_2\xi_1 & e^{2d_1}M_3\xi_1^2 + L_2\xi_2^2 + e^{2d_3}M_1\xi_3^2 - \Omega^2 & N_1\xi_2\xi_3 \\ N_2\xi_3\xi_1 & N_1\xi_3\xi_2 & e^{2d_1}M_2\xi_1^2 + e^{2d_2}M_1\xi_2^2 + L_3\xi_3^2 - \Omega^2 \end{vmatrix}$$

We recall that $\Omega = \omega + u_1\xi_1 + u_2\xi_2 + u_3\xi_3$.

The condition of hyperbolicity of the system requires the matrix to be positive definite, the roots of the matrix being Ω^2. In particular, for arbitrary ξ_1, ξ_2, ξ_3 not all zero, the sum $L_1\xi_1^2 + e^{2d_2}M_3\xi_2^2 + e^{2d_3}M_2\xi_3^2$ must be positive, i.e., for hyperbolicity L_1, M_2, M_3 must be positive. We recall that in Sec. 3 the inequalities

$$\frac{a_2E_{a_2} - a_3E_{a_3}}{a_2^2 - a_3^2} = \left(\frac{\rho}{\rho_0}\right)^{2/3} M_1 > 0, \quad \frac{a_3E_{a_3} - a_1E_{a_1}}{a_3^2 - a_1^2} = \left(\frac{\rho}{\rho_0}\right)^{2/3} M_2 > 0$$
$$\frac{a_1E_{a_1} - a_2E_{a_2}}{a_1^2 - a_2^2} = \left(\frac{\rho}{\rho_0}\right)^{2/3} M_3 > 0$$

guaranteed satisfaction of the law of growth of entropy for relaxation of the tangential stresses. In the linear elasticity theory case it is assumed that the deformations are small, i.e., in every case d_1, d_2, d_3 may be regarded as small ($e^{2di} \approx 1$, $\rho \approx \rho_0$), and the equation of state is given in the form

$$\rho_0 E = {}^1/_2\lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 + \mu(\varepsilon_{ij}\varepsilon_{ji}) = (\lambda/2 + \mu/3)(\varepsilon_1 + \\ + \varepsilon_2 + \varepsilon_3)^2 + \mu[(\varepsilon_1 - {}^1/_3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))^2 + (\varepsilon_2 - {}^1/_3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))^2 + \\ + (\varepsilon_3 - {}^1/_3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3))^2] \approx (\lambda/2 + \mu/3)(1 - \rho/\rho_0)^2 + 2\mu D$$

It is then found that

$$L_1 = L_2 = L_3 = (\lambda + 2\mu)/\rho_0, \qquad M_1 = M_2 = M_3 = \mu/\rho_0$$
$$N_1 = N_2 = N_3 = (\lambda + \mu)/\rho_0$$

and the characteristic equation appears in the form

$$\left[\Omega^2 - \frac{\lambda + 2\mu}{\rho_0}(\xi_1^2 + \xi_2^2 + \xi_3^2)\right]\left[\Omega^2 - \frac{\mu}{\rho_0}(\xi_1^2 + \xi_2^2 + \xi_3^2)\right]\left[\Omega^2 - \frac{\mu}{\rho_0}(\xi_1^2 + \xi_2^2 + \xi_3^2)\right]^2 = 0$$

This form of the characteristic equation is known from the linear theory of elasticity.

We now proceed to calculate the matrices A^{ij}, i.e., to obtain the Eqs. (5.1). From Eqs. (4.11) of Sec. 4 it follows that to calculate the A^{ij}_{kl} it is sufficient to calculate the first and second derivatives of E with respect to the deformation tensor components, namely, $E_{\varepsilon_{ij}}$, $E_{\varepsilon_{ij}\varepsilon_{kl}}$.

We calculate the derivatives in the principal axes of the deformation tensor, i.e., for $\varepsilon_{ij} = 0$ ($i \neq j$), $\varepsilon_{ii} = \varepsilon_{ii}^\circ$. Let ε_i be the roots of the characteristic equation

$$\begin{vmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon \end{vmatrix} = -(\varepsilon - \varepsilon_1)(\varepsilon - \varepsilon_2)(\varepsilon - \varepsilon_3) = 0$$

We assume that none of the diagonal elements coincide. This restriction can be removed later by a continuous extension of the equations for the A^{ij}_{kl} to the case of one multiplicity or another. We expand ε_1, ε_2, ε_3 in powers of $\varepsilon_{ii} - \varepsilon_{ii}^\circ$ ($i \neq j$)

$$\varepsilon_i = \varepsilon_{ii}^\circ + (\varepsilon_{ii} - \varepsilon_{ii}^\circ) + \sum_{p \neq q,\, r \neq s} \alpha^i_{pqrs}\varepsilon_{pq}\varepsilon_{rs} + \ldots = \varepsilon_{ii} + \delta_i$$

To calculate α^1_{pqrs} we substitute this expansion into the characteristic equation

$$\begin{vmatrix} -\delta_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_{11} - \delta_1 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_{11} - \delta_1 \end{vmatrix} = -\delta_1(\varepsilon_{22} - \varepsilon_{11})(\varepsilon_{33} - \varepsilon_{11}) - (\varepsilon_{33} - \varepsilon_{11})\varepsilon_{12}\varepsilon_{21} - (\varepsilon_{22} - \varepsilon_{11})\varepsilon_{13}\varepsilon_{31} + \ldots = 0$$

The quantity δ_i is of the second order of smallness and the terms discarded are of the fourth order of smallness.

From this we have

$$\delta_1 = -\frac{\varepsilon_{12}\varepsilon_{21}}{\varepsilon_{22} - \varepsilon_{11}} - \frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33} - \varepsilon_{11}} + \ldots = -\frac{\varepsilon_{12}\varepsilon_{21}}{\varepsilon_{22}^\circ - \varepsilon_{11}^\circ} - \frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}^\circ - \varepsilon_{11}^\circ} + \ldots$$
$$\varepsilon_1 = \varepsilon_{11} - \frac{\varepsilon_{12}\varepsilon_{21}}{\varepsilon_{22}^\circ - \varepsilon_{11}^\circ} - \frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}^\circ - \varepsilon_{11}^\circ}$$

and, further,

$$\frac{\partial \varepsilon_1}{\partial \varepsilon_{11}} = 1, \qquad \frac{\partial \varepsilon_1}{\partial \varepsilon_{pq}} = 0$$

if, simultaneously, we do not have the condition $p = 1$, $q = 1$

$$\frac{\partial^2 \varepsilon_1}{(\partial \varepsilon_{ij})^2} = 0, \quad \frac{\partial^2 \varepsilon_1}{\partial \varepsilon_{23}\partial \varepsilon_{32}} = 0, \quad \frac{\partial^2 \varepsilon_1}{\partial \varepsilon_{12}\partial \varepsilon_{21}} = \frac{1}{\varepsilon_{11}^\circ - \varepsilon_{22}^\circ} = \frac{1}{\varepsilon_1 - \varepsilon_2}$$
$$\frac{\partial^2 \varepsilon_1}{\partial \varepsilon_{13}\partial \varepsilon_{31}} = \frac{1}{\varepsilon_{11}^\circ - \varepsilon_{33}^\circ} = \frac{1}{\varepsilon_1 - \varepsilon_3}$$

All the remaining derivatives are calculated in an analogous way, being obtained by cyclically permuting the subscripts.

As a result, we have

$$E_{\varepsilon_{pq}} = E_{\varepsilon_i} \frac{\partial \varepsilon_i}{\partial \varepsilon_{pq}}$$

$$E_{\varepsilon_{pq}\varepsilon_{rs}} = E_{\varepsilon_i} \frac{\partial^2 \varepsilon_i}{\partial \varepsilon_{pq} \partial \varepsilon_{rs}} + E_{\varepsilon_i \varepsilon_j} \frac{\partial \varepsilon_i}{\partial \varepsilon_{pq}} \frac{\partial \varepsilon_j}{\partial \varepsilon_{rs}}$$

$$E_{\varepsilon_{ii}} = E_{\varepsilon_i}, \qquad E_{\varepsilon_{ii}\varepsilon_{jj}} = E_{\varepsilon_i \varepsilon_j}$$

$$E_{\varepsilon_{ij}\varepsilon_{ji}} = \frac{E_{\varepsilon_i} - E_{\varepsilon_j}}{\varepsilon_i - \varepsilon_j} \qquad (i \neq j)$$

and all the remaining derivatives are zero.

We consider the parametrization

$$E = E(a_1, a_2, a_3, S) = E\left(\frac{1}{\sqrt{1-2\varepsilon_1}}, \frac{1}{\sqrt{1-2\varepsilon_2}}, \frac{1}{\sqrt{1-2\varepsilon_3}}, S\right)$$

In terms of the a_i the derivatives may be written

$$\begin{gathered} E_{\varepsilon_{ii}} = a_i^3 E_{a_i}, \qquad E_{\varepsilon_{ii}\varepsilon_{jj}} = a_i^3 a_j^3 E_{a_i a_j} \\ E_{\varepsilon_{ii}\varepsilon_{ii}} = a_i^6 E_{a_i a_i} + 3a_i^5 E_{a_i} \\ E_{\varepsilon_{ij}\varepsilon_{ji}} = 2a_i^2 a_j^2 \frac{a_i^3 E_{a_i} - a_j^3 E_{a_j}}{a_j^2 - a_i^2} \ (i \neq j) \end{gathered} \tag{5.3}$$

and all the remaining derivatives are zero.

The use of Eqs. (5.3) in Eqs. (4.11) leads to the Eqs. (5.1) for the A^{ij}_{kl} in the coordinate system chosen. Obtaining Eqs. (5.2) from Eqs. (5.1) is an elementary, even though somewhat involved, exercise in permuting the variables. We shall not supply the details here.

6. Symmetric System of Equations of the First Order. In Sec. 4 it was shown how the equations of elasticity theory can be reduced to the form (4.10).

This system can be rewritten as

$$\left(\frac{d}{dt}\right)^2 u - (A^{ij} + X^{ij}) \frac{\partial^2 u}{\partial x_i \partial x_j} - B^i \frac{\partial v}{\partial x_i} = F$$

$$\frac{dv}{dt} = C^k \frac{\partial u}{\partial x_k} + \varphi$$

In this way of writing the equations we have introduced completely arbitrary skew-symmetric ($X^{ij} = -X^{ji} = -X^{ij*}$) matrices, concrete expressions for which are given later.

We introduce the new variables

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} du_1/dt \\ du_2/dt \\ du_3/dt \end{pmatrix}, \qquad q_j = \begin{pmatrix} q_{1j} \\ q_{2j} \\ q_{3j} \end{pmatrix} = \begin{pmatrix} \partial u_1/\partial x_j \\ \partial u_2/\partial x_j \\ \partial u_3/\partial x_j \end{pmatrix}$$

and with their aid we rewrite the equations in the form of a first order system:

$$\begin{gathered} du/dt = w \\ \frac{dw}{dt} - (A^{ij} + X^{ij}) \frac{\partial q_j}{\partial x_i} - B^i \frac{\partial v}{\partial x_i} = F \\ \frac{dq_j}{dt} - \frac{\partial w}{\partial x_j} = -q_{\alpha j} q_\alpha \quad (j = 1, 2, 3) \\ dv/dt = \psi \end{gathered}$$

Instead of this system, it is convenient to consider the system obtained by replacing the equations in the last two rows by linear combinations of them

$$
\begin{gathered}
du \,/\, dt = w \\
\frac{dw}{dt} - (A^{ij} + X^{ij}) \frac{\partial q_j}{\partial x_i} - B^i \frac{\partial v}{\partial x_i} = F \\
(A^{ji} + X^{ji}) \frac{dq_i}{dt} + B^j \frac{dv}{dt} - (A^{ji} + X^{ji}) \frac{\partial w}{\partial x_i} = \Phi_j \quad (j = 1, 2, 3) \\
B^{k*} \frac{dq_k}{dt} + P \frac{dv}{dt} - B^{k*} \frac{\partial w}{\partial x_k} = \Psi
\end{gathered}
$$

In the last equation we have introduced yet another arbitrary matrix P; how it is to be selected will be described later. If we introduce the vector of unknown functions

$$
U = \begin{pmatrix} u \\ w \\ q_1 \\ q_2 \\ q_3 \\ v \end{pmatrix}
$$

we can write the latter system in the following symmetric form:

$$
\begin{pmatrix}
I_3 & 0 & 0 & 0 & 0 & 0 \\
0 & I_3 & 0 & 0 & 0 & 0 \\
0 & 0 & A^{11} & A^{12} + X^{12} & A^{13} + X^{13} & B^1 \\
0 & 0 & A^{21} + X^{21} & A^{22} & A^{23} + X^{23} & B^2 \\
0 & 0 & A^{31} + X^{31} & A^{32} + X^{32} & A^{33} & B^3 \\
0 & 0 & B^{1*} & B^{2*} & B^{3*} & P
\end{pmatrix} \frac{dU}{dt} +
$$

$$
+ \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -A^{i1} - X^{i1} & -A^{i2} - X^{i2} & -A^{i3} - X^{i3} & -B^i \\
0 & -A^{1i} - X^{1i} & 0 & 0 & 0 & 0 \\
0 & -A^{2i} - X^{2i} & 0 & 0 & 0 & 0 \\
0 & -A^{3i} - X^{3i} & 0 & 0 & 0 & 0 \\
0 & -B^{i*} & 0 & 0 & 0 & 0
\end{pmatrix} \frac{\partial U}{\partial x_i} = \begin{pmatrix} w \\ F \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Psi \end{pmatrix} \tag{6.1}
$$

In order for this system to be a symmetric t-hyperbolic system in the Friedrichs sense, it is necessary to choose the X^{ij} and P so that the matrix coefficient of dU/dt will be positive definite. It is not difficult to show that if the matrix

$$
\begin{pmatrix}
A^{11} & A^{12} + X^{22} & B^{13} + X^{13} \\
A^{21} + X^{21} & A^{22} & A^{23} + X^{23} \\
A^{31} + X^{31} & A^{32} + X^{32} & A^{33}
\end{pmatrix} \tag{6.2}
$$

is positive definite, then by choosing the matrix P to be "sufficiently large" (i.e., a positive definite P with its least characteristic value sufficiently large) we can assure positive definiteness of the entire matrix of interest. For the X^{ij} we take

$$
X_{kl}{}^{ij} = \begin{cases} A_{kl}{}^{ij}, & \text{if} \quad k > l,\ i < j \\ 0\ , & \text{if} \quad k = l,\ i \neq j \\ -A_{kl}{}^{ij}, & \text{if} \quad k < l,\ i < j \end{cases}
\qquad X_{kl}{}^{ii} = 0
$$

With this choice of X^{ij} the matrix (6.2) appears as follows in a system of coordinates with axes directed along the principal axes of the deformation tensor:

$$\begin{pmatrix} L_1 & 0 & 0 & 0 & N_3 & 0 & 0 & 0 & N_2 \\ 0 & e^{2d_1}M_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2d_1}M_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2d_2}M_3 & 0 & 0 & 0 & 0 & 0 \\ N_3 & 0 & 0 & 0 & L_2 & 0 & 0 & 0 & N_1 \\ 0 & 0 & 0 & 0 & 0 & e^{2d_2}M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{2d_3}M_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2d_3}M_1 & 0 \\ N_2 & 0 & 0 & 0 & N_1 & 0 & 0 & 0 & L_3 \end{pmatrix}$$

The conditions for it to be positive definite are the following:

$$M_1 > 0, \quad M_2 > 0, \quad M_3 > 0 \tag{6.3}$$
$$L_1 > 0, \quad L_1L_2 - N_3^2 > 0, \quad \begin{vmatrix} L_1 & N_3 & N_2 \\ N_3 & L_2 & N_1 \\ N_2 & N_1 & L_3 \end{vmatrix} > 0$$

These conditions represent restrictions on the equation of state

$$E = E(\rho, D, \Delta, S)$$

It is interesting to note that these restrictions are more stringent than the conditions for hyperbolicity of the system (4.10); finally, we note that they depend on the specific choice of the X^{ij}.

As we have already remarked (see Sec. 5), in the case of the linear theory of elasticity

$$L_1 = L_2 = L_3 = (\lambda + 2\mu) / \rho_0$$
$$M_1 = M_2 = M_3 = \mu / \rho_0$$
$$N_1 = N_2 = N_3 = (\lambda + \mu) / \rho_0$$

The conditions (6.3) for positive definiteness then reduce to the inequalities

$$\mu > 0, \ \lambda > -4/3\,\mu$$

while the conditions for hyperbolicity are

$$\mu > 0, \ \lambda > -2\,\mu$$

It is noted in [16] that, as a rule, for all elastic media

$$\mu > 0, \quad \lambda > 0$$

We remark that the roots of the characteristic determinant of the system (6.1), namely,

$$\begin{vmatrix} \Omega I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega I_3 & -(A^{i1}+X^{i1})\xi_i & -(A^{i2}+X^{i2})\xi_i & -(A^{i3}+X^{i3})\xi_i & -B^i\xi_i \\ 0 & -(A^{1i}+X^{1i})\xi_i & \Omega(A^{11}+X^{11}) & \Omega(A^{12}+X^{12}) & \Omega(A^{13}+X^{13}) & \Omega B^1 \\ 0 & -(A^{2i}+X^{2i})\xi_i & \Omega(A^{21}+X^{21}) & \Omega(A^{22}+X^{22}) & \Omega(A^{23}+X^{23}) & \Omega B^2 \\ 0 & -(A^{3i}+X^{3i})\xi_i & \Omega(A^{31}+X^{31}) & \Omega(A^{32}+X^{32}) & \Omega(A^{33}+X^{33}) & \Omega B^3 \\ 0 & -B^{i*}\xi_i & \Omega B^{1*} & \Omega B^{2*} & \Omega B^{3*} & \Omega P \end{vmatrix} =$$
$$= \Omega^{16} \det\|\Omega^2 I_3 - \xi_i\xi_j A^{ij}\| \det \begin{pmatrix} A^{11}+X^{11} & A^{12}+X^{12} & A^{13}+X^{13} & B^1 \\ A^{21}+X^{21} & A^{22}+X^{22} & A^{23}+X^{23} & B^2 \\ A^{31}+X^{31} & A^{32}+X^{32} & A^{33}+X^{33} & B^3 \\ B^{1*} & B^{2*} & B^{3*} & P \end{pmatrix}$$

do not depend on the matrices X^{ij}, B^i, P.

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